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THE HEAT GENERATED DURING THE TORSIONAL OSCILLATIONS OF COPPER TUBES

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Abstract—Experimental data on the heat generated during slow torsional oscillations of annealed OFHC copper tubes are presented. These oscillations involve a strain amplitude of approximately $\frac{1}{2}$ per cent so that plastic deformations occur. The frequency of oscillation is less than $\frac{1}{4}$ c/s. It is demonstrated that there is a period when nearly adiabatic conditions exist and all observations are made in this time. It is demonstrated that the rate of doing plastic work sometimes exceeds the rate of heat generation and vice-versa. Over a complete cycle the work always exceeds the heat that is generated, but locally this is not the case. Specimens, which are releasing more heat than the plastic work that is being done on them, have unsymmetrical hysteresis loops and sometimes deform nonhomogeneously in the first cycle. All specimens require more plastic work than they release as heat when they leave the elastic region in the first cycle.

A coupled thermoplasticity theory which is consistent with the experimental data is also presented in an appendix.

1. INTRODUCTION

General

WHEN a solid body is deformed, the temperature is also changed because the frequency of atomic vibrations is altered due to the variation of the interatomic distances. This effect manifests itself in a continuum mechanics theory known as coupled thermoelasticity. For slow processes and when the material response remains elastic, the temperature changes are very small and isothermal conditions are usually justified. The linear coupled thermoelastic theory that is normally used predicts no temperature change in shearing motions but heating in compression and cooling in tension is considered. Thus cyclic deformations produce no net increase in temperature. In the inelastic range of the material response it is likely that much larger temperature changes will exist during the deformation because energy is also dissipated. For example, it is likely that the temperature in the vicinity of the slip plane during plastic deformation is changed. That these local temperature changes result in observable macroscopic temperature increases has been demonstrated [1–3]. The metallurgical literature abounds [2, 4, 5] in data in which the "stored energy of cold work" is measured. This quantity is the *difference* between the work done on the specimen and the heat generated. It is approximately equal to 10 per cent of the work done on the specimen and is used as a measure of the instability of metallurgical variables.

The experimental phase of the present paper concentrates on the study of the *heat generated* in order to ultimately develop a theory along the lines of linear coupled thermoelasticity. As a secondary result it provides data on the *rate* of heat generation and rate of doing plastic work. It is found that the plastic work done on the specimen per cycle always equals or exceeds the heat generated during the cycle. However, this is not true at a particular instant of time. This should be of value in understanding the defect structure and the ultimate mechanisms responsible for plastic behavior. The particular deformations that are used are slow torsional oscillations which result in strains of less than $\frac{1}{2}$ of 1 per cent. We have endeavored to minimize bending strains and have verified that the axial strains are (experimentally) small compared to the shearing deformations.

A theoretical thermoplasticity theory which generalizes [1] and which is in better agreement with the present experimental data is also presented.

Technique

The measurement of the temperature changes during deformation is straightforward in principle but is not commonly done in solid mechanics. The author recently developed a technique [1, 6] which makes this measurement routine in metals. It is sensitive enough to measure the coupled thermoelastic effect in aluminum due to a change of 100 psi in tension. Basically, the method is the same as reported in [1] but improvements described in [6] permit the simultaneous measurement of stress, strain and temperature with less noise than before. The specimen is mounted in grips which are electrically floated with respect to the measurement system. A thermocouple is made by attaching two wires of dissimilar materials to the surface of the specimen. The wire tips are located near one another and are held in place by electrical tape. The thermocouple output is added to a voltage of polarity opposite to that existing in the undeformed configuration. The result of this addition is amplified, by 10^5, and recorded on an oscilloscope (by a camera) with a four-channel plug-in unit. Considerable technique is required [6] to produce a good signal-to-noise ratio but this can be done. The new method can measure more rapidly changing temperatures than the one reported in [1].

Most of the heat lost is by conduction to the grips and to regions which are not highly strained. There exists a period of time (20 sec) in which the heat lost by conduction is small compared to that which is generated. This time is experimentally determined but is in agreement with the analysis of Taylor and Farren [3] adapted to the present situation. During this initial time, just after the motor is started, the conditions are approximately adiabatic no matter how slow the motor runs.

Stress and strain measures

The strength of materials stress measure is used for presenting the experimental results. The average shearing stress σ_{23} is related to the applied torque T by

$$\sigma_{23} = T/2\pi r^2 h \tag{1}$$

where h is the thickness of the tube and r is the mean radius. Equation (1) assumes that the oscillations are slow enough that mechanical equilibrium relations can be used.

The deformation used in all tests is the oscillation of one end ($x_3 = 0$) of a tube through an angle of amplitude φ while the other end ($x_3 = L$) is fixed against rotation. Twice the average shearing strain ε_{23} is the strength of materials strain γ given by

$$\gamma = 2\varepsilon_{23}\ (\text{average}) = \varphi r/L \tag{2}$$

where L is the length of the specimen. The local value of ε_{23} is measured by SR4 strain gages. It is experimentally found that some specimens deform so that the strain is nonhomogeneous along the axis. This is especially important in the very first cycle of the deformation of annealed specimens. The axial strain ε_{33} is experimentally determined to be small compared to ε_{23}. By using two axial gages it has been verified that bending deformations are also small. It is found that, as in aluminum, the axial strain does not vary as the square of the shearing strain but is more nearly linear in this variable.

The end ($x_3 = L$) is fixed against rotation but axial motion is freely permitted by mounting that end on wheels. Thus the axial stress σ_{33} should be very small.

Energy equation

The underlying purpose of this investigation is to establish more knowledge of the energy equation during plastic deformation. The energy equation for a solid that is conducting heat is

$$\rho_0 \dot{U} = \sigma_{i\alpha}\dot{\varepsilon}_{i\alpha} - \partial Q_\alpha / \partial X_\alpha \tag{3}$$

where Cartesian tensors and the usual summation convention of repeated indices is used; where Q_α is the heat flux vector, X_α is the position of a generic particle, U is the internal energy per unit volume, and ρ_0 is the initial density of the material. In equation (3), $\sigma_{i\alpha}$ and $\varepsilon_{i\alpha}$ are the components of the stress and small strain tensors respectively. Because the axial strain ε_{33} is known to be small, the use of the small strain tensor in equation (3) is legitimate in our experiments.

The first term on the right-hand side of equation (3) is the rate of doing work on an element of the body. In the plastic range of the response, some energy is recovered on unloading and purely dilational deformations are assumed to remain elastic. Under these conditions it is convenient to consider the rate of doing plastic work $\dot{\bar{U}}^*$ defined by

$$\dot{\bar{U}}^* = S_{i\alpha}(\dot{e}_{i\alpha} - \dot{S}_{i\alpha}/2\mu) \tag{4}$$

where superimposed dots are derivatives with respect to time, μ is Lamé's elastic constant, $S_{i\alpha}$ is the stress deviator, and $e_{i\alpha}$ is the strain deviator. These latter quantities are defined as

$$S_{i\alpha} = \sigma_{i\alpha} - \sigma_{kk}\delta_{i\alpha}/3$$

and

$$e_{i\alpha} = \varepsilon_{i\alpha} - \varepsilon_{kk}\delta_{i\alpha}/3$$

where $\delta_{i\alpha}$ is the Kronecker delta symbol. For the case of torsional oscillations described above, equation (4) becomes

$$\dot{\bar{U}}^* = 2\sigma_{23}(\dot{\varepsilon}_{23} - \dot{\sigma}_{23}/2\mu) \tag{5}$$

The plastic work that is done when the strain changes between $\bar{\varepsilon}_{23}$ and ε_{23} is

$$\bar{U}^* = 2\int_{\bar{\varepsilon}_{23}}^{\varepsilon_{23}} \sigma_{23}(\mathrm{d}\varepsilon_{23} - \mathrm{d}\sigma_{23}/2\mu) \tag{6}$$

which is evaluated by using the experimental $\sigma_{23}(\varepsilon_{23})$ relation and small increments of strain.

It is a basic assumption of plasticity theory; which at least in some circumstances [1], can be derived from the thermodynamic postulate of positive entropy production, that

$$\dot{\bar{U}}^* \geq 0 \tag{7}$$

This condition has been experimentally verified when equation (5) is used.

Most of the experimental data discussed below are taken in an approximately adiabatic situation so that Q_α vanishes in equations (3). Furthermore, under the experimental conditions of unloading, the temperature usually remains constant. This means that the dilational contributions to the internal energy are small in torsional oscillations.

From the results of the Appendix, the difference between the rate of doing plastic work and the rate of heat generation is

$$(\rho_0 \partial U/\partial e''_{i\alpha})\dot{e}''_{i\alpha} = S_{i\alpha}\dot{e}''_{i\alpha} - \rho_0 C_D \dot{\theta} \tag{8}$$

where C_D is the specific heat at constant deformation and θ is the temperature difference above a reference value, and $e''_{i\alpha}$ is the plastic strain. At the beginning of this investigation it was presumed that the right-hand side of equation (8) would be non-negative. This has *not* always been observed. However, it has been found that

$$\oint S_{ij}\dot{e}''_{ij}\,dt \geq \rho_0 C_D \oint \dot{\theta}\,dt \tag{9}$$

when the integrals are taken over a *complete* cycle in strain and where

$$\rho_0 C_D = 300 \text{ in-lb/in}^3/^\circ\text{F}$$

is used for copper. The order of the discrepancy between the two sides of equation (9) is 10 per cent and is in agreement with [3]. The data are presented below and will be further discussed later. It is possibly significant that many of the procedures described in [2] and [7] correspond to evaluating the integral of equation (8) between two zero stress states (or between other conditions) but not equation (8) itself.

2. EXPERIMENTAL DATA

Material

OFHC copper tubes were used in all of the tests reported here. These tubes were made by a commercial drawing process and were purchased as a single order in order to minimize variations in properties. Thus the variations between specimens that will be described were surprising to the author. Phase changes do not occur near room temperature [7] in copper and any tendency to recrystallize would have to vary considerably from specimen to specimen and be randomly distributed with respect to time. Most of the specimens were annealed at 1100°F for 2 hr and furnace cooled. They were imbedded in charcoal during annealing to reduce the absorption of oxygen. Again the variation in response occurred when the specimens were annealed in the same batch. The tubes were $\frac{1}{4}$ in. o.d. with a $\frac{1}{32}$-in. wall thickness and were 8·5 in. long unless noted otherwise. Approximately 100 specimens were tested but only a few are reported in this paper.

The "static" stress–strain relation for the annealed material is shown in Fig. 1. These data were obtained by loading the specimen in a dead weight apparatus until the over-all angle was approximately the same as used in some of the oscillatory tests. This static test apparatus did not permit axial displacements of the end $X = L$, as the oscillatory equipment does. In aluminum this extra constraint is not important and there is no indication that it is in copper. The capital C's in Fig. 1 refer to the "creep" that takes place between load increments in the static test. It is noteworthy that the variation in material response between specimens in the static test is much less than the slowly oscillated specimens. The variation in static test results is quite normal in the author's limited experience. However, [8] apparently observed a wide variation in the static test results.

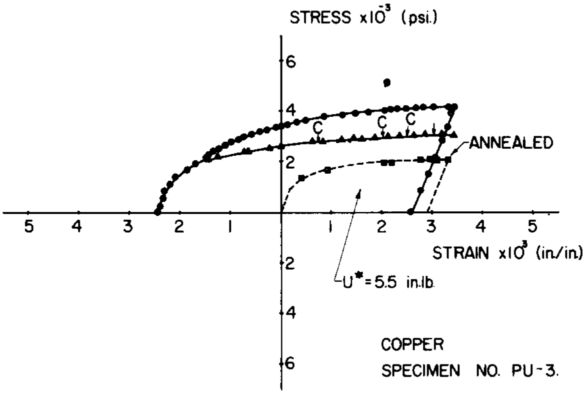

FIG. 1. The average shearing stress versus average shearing strain γ for annealed copper. The test was made in a dead weight torsion apparatus. Note: The "C" 's denote deformation occurring as creep and not immediately upon load application.

The remaining curves in Fig. 1 were obtained by twisting the specimen in the reverse direction to a strain of $-0{\cdot}0035$ and then releasing the load. The specimen was then reloaded with a positive stress as indicated in Fig. 1. No data were taken during the negative stress regions of the static test.

Technique

The specimens are slowly oscillated in torsion in such a way that a prescribed over-all angular displacement of the end $X_3 = 0$ is imposed. The applied torque (force), strain at a point (points), and temperature are simultaneously measured as a function of time. This simultaneous measurement is an improvement over the method in [1]. The temperature reported is that existing at the center of the specimen. After a number of cycles (say four), the motor is turned off and the specimen allowed to cool and equilibrate its internal motion. Usually 15 min is allowed between runs. When the motor is stopped, the over-all angle is maintained constant but the local strain and the stress can, and do, redistribute themselves. Although very large temperature changes can be produced by this method at high frequency and by not stopping, the data reported here were at temperatures very close to room temperature in order to minimize structural changes.

A typical response for the first cycle on an annealed specimen (No. TT-5) is shown in Fig. 2. Note for future reference that the local strain, measured near the center of the specimen, is less in the positive direction than it is in the negative, even though the over-all angle is the same in both directions. This illustrates the axial variation in strain which frequently exists during the first cycle. Note that the maximum stress is very nearly the same as the static results shown in Fig. 1. This is not always the case. Note also that the temperature remains approximately constant when the specimen "unloads".

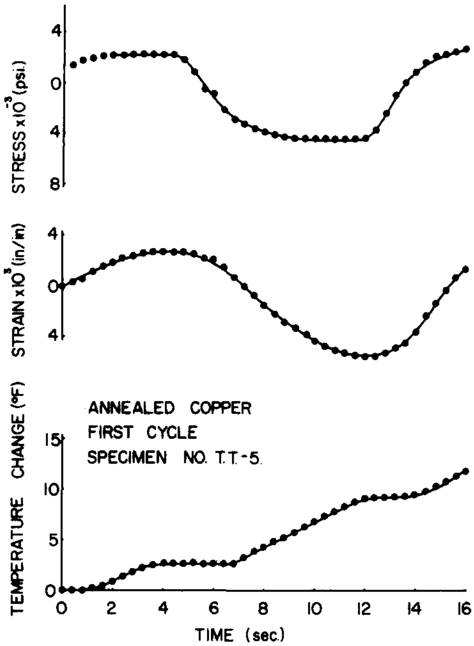

FIG. 2. The stress, center strain $2\varepsilon_{23}(L/2)$, and temperature as a function of time for annealed copper specimen No. TT-5 subjected to a torsional oscillation of $\gamma = \pm 0{\cdot}0045$.

Derived data

The data shown in Fig. 2 are considered to be of sufficient quality that one can rely on the time derivatives of the functions. This has been done with the results for the temperature shown in Fig. 3(a) as solid lines. Also presented in Fig. 3(a) is the rate of doing plastic work, equation (5). It is seen in this figure that the rate of temperature increase exceeds the rate of doing work and vice-versa. It will be shown below that this does not always happen but that it occurs sufficiently often that it is real. In every specimen, the first cycle was such that at small times, the rate of doing plastic work exceeded the rate of heat generation. Note that the area under the curves for the cycle (i.e. to 13 sec) are such that slightly more plastic work is done than appears as heat, as mentioned in equation (9).

Data on $\dot{\theta}$ and $\dot{U}^*$ for the fifth cycle of this same specimen are shown in Fig. 3(b). After taking the data for Fig. 3(a), the motor was stopped and the specimen allowed to cool prior to taking the data for Fig. 3(b).

Another useful form for part of the data shown in Fig. 2 is the hysteresis loop that is developed. The first loop for specimen No. TT-5 is shown as a dotted curve in Fig. 4. The hysteresis loop for the fifth cycle is also shown in Fig. 4 as a solid line. Note that the stress in the fifth cycle is less than in the first. This is a result of some relaxation having occurred during the period when the motor is off.

Other specimens

Another example of the response which is in the same category as specimen No. TT-5 is shown in Figs. 5–7. The temperature change and stress are shown in Fig. 5 for

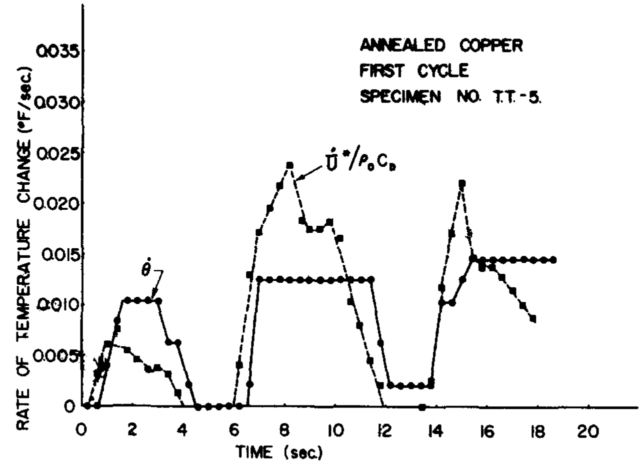

FIG. 3(a). The rate of temperature change $\dot{\theta}$ and the rate of doing plastic work $\dot{U}^*/\rho_0 C_D$. These results obtained from the data shown in Fig. 2.

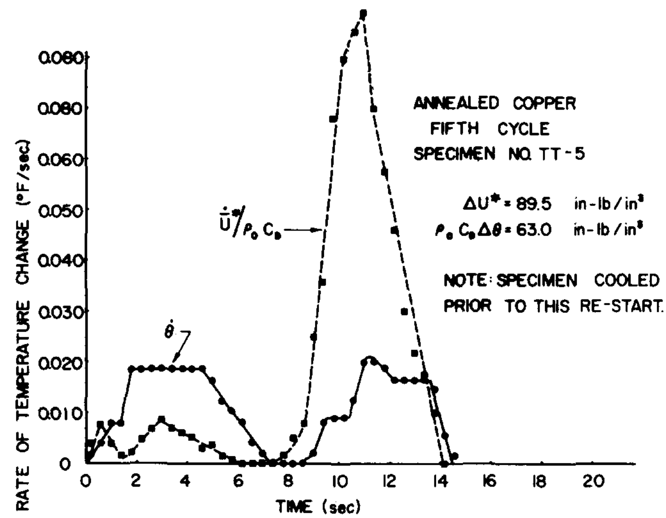

FIG. 3(b). The rate of temperature increase $\dot{\theta}$ and the rate of doing plastic work $\dot{U}^*/\rho_0 C_D$ for the fifth cycle of the specimen used for Fig. 3(a).

specimen No. A-6A. The maximum stress in the first cycle is much less than in Figs. 1 and 2. The nonhomogeneous strain field is explicit in this specimen as shown in Fig. 6. The two gages respond in precisely the same manner after the first 10 sec. During many subsequent cycles they also responded in an identical manner. Thus there was a nonhomogeneous shifting of the reference configuration for strain which, when completed, remained constant in time. Notice in Fig. 5 that at 3 sec and again at 14 sec there is local cooling even though plastic work is being done on the entire specimen. This is *not* normal cooling such as radiation or conduction through the grips but is related to the mechanism which is responsible for the nonhomogeneous strain shown in Fig. 6. The hysteresis

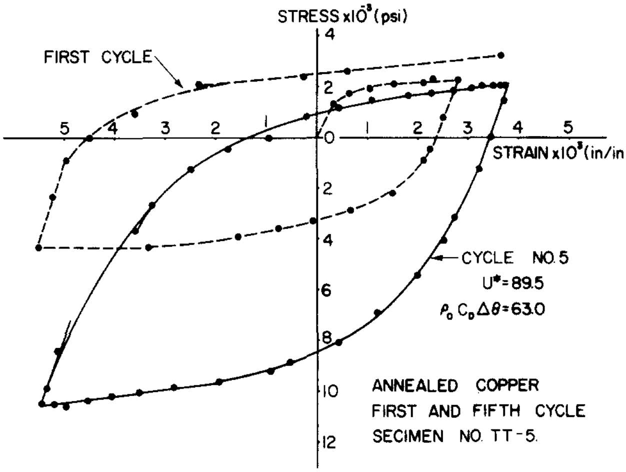

FIG. 4. The hysteresis loop for annealed copper specimen No. TT-5 during the first and fifth cycles. The strain measure is twice the value of the strain measured at the center, $2\varepsilon_{23}(x_3 = L/2)$.

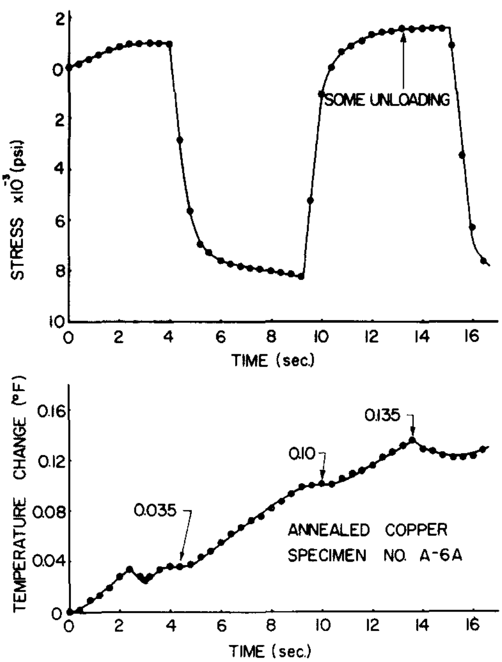

FIG. 5. The stress and center temperature rise in annealed copper specimen No. A-6A. This response is not typical. The amplitude of the oscillation is $\gamma = 0{\cdot}00368$ in/in.

loop for this specimen is shown in Fig. 7 where the strain used is the front gage. As indicated in Fig. 7, the heat generated on this basis in the first quarter of a cycle is

$$\rho_0 C_D \Delta\theta = 10{\cdot}5 \text{ in-lb/in}^3$$

while only $1{\cdot}44$ in-lb/in^3 of plastic work is done. Subsequent cycles of this specimen did not show any decreasing temperature such as those shown in Fig. 5.

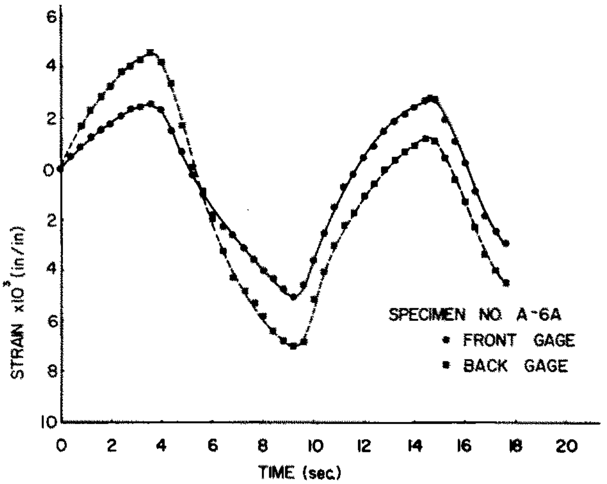

FIG. 6. Strain ($2\varepsilon_{23}$) versus time for annealed copper specimen No. A-6A as given by gages located at two points. The stress and center temperature history for this specimen is shown in Fig. 5. The front gage was located 1·5 in. toward the twisted end ($x_3 = 0$) from the center. The back gage was located 1·5 in. toward the fixed end ($x_3 = L$) from the specimen center.

A third example of this type of response is shown in Figs. 8 and 9 for specimen AG-22-G. The data shown in Fig. 8 are for the thirty-fifth cycle of this specimen and are not particularly typical data. This specimen has the following unusual responses. As indicated in Fig. 9, during the first cycle the stress suddenly decreased. As shown in Fig. 8 the temperature continued to increase when the specimen was unloading, although at a reduced rate. As shown in Fig. 9, the strain amplitude at the center is less than it was in the first cycle. Even after this number of cycles, the temperature sometimes rises at a rate faster than the rate of doing plastic work. Thus this specimen is still attempting to reach a more stable (lower energy) state. The heating during unloading is not commonly observed, and has only been observed in those specimens which deform nonhomogeneously in their first quarter cycle.

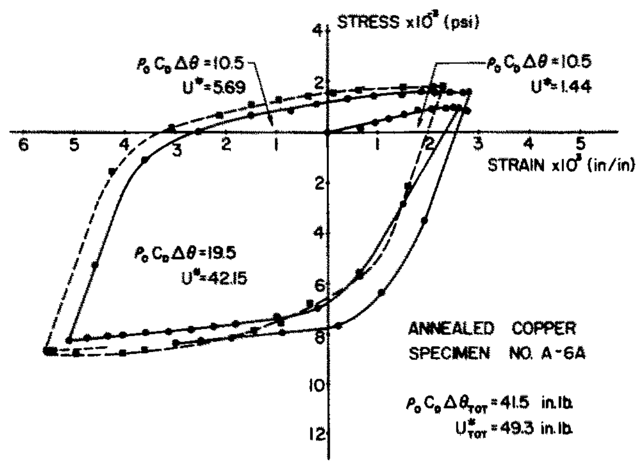

FIG. 7. Hysteresis loops for specimen No. A-6A for which data are also presented in Figs. 5 and 6. The strain measure is twice ε_{23} of the front gage.

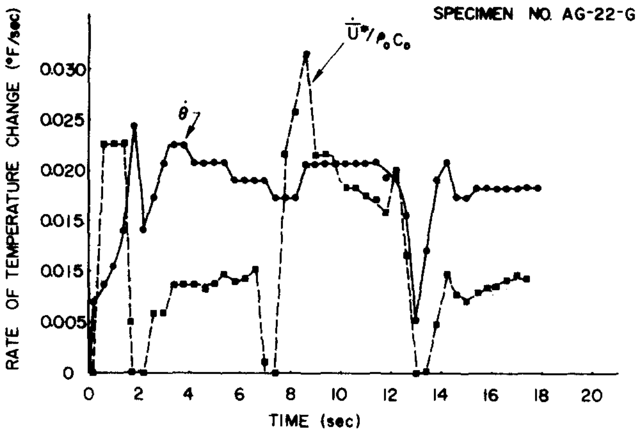

FIG. 8. The rate of temperature increase and the rate of doing plastic work for specimen No. AG-22-G after thirty-five cycles with several stops. Note that the temperature rate $\dot{\theta}$ does not go to zero when the rate of doing plastic work does.

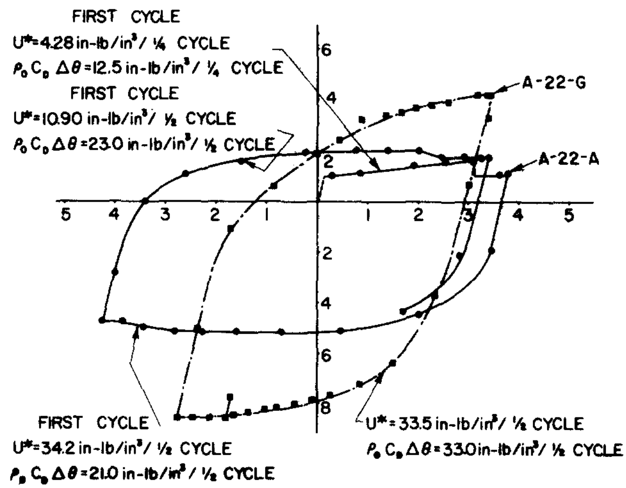

FIG. 9. Hysteresis loop for specimen No. AG-22-G for which other data are presented in Fig. 8. Note the sudden decrease in stress in the first quarter cycle. The amplitude of the oscillation is 0·00368.

A different response

In some specimens, the first quarter cycle response is different from those shown in Figs. 2, 5, and 8. A typical example of this second type of response is shown in Figs. 10 and 11(a) for specimen No. AG-1. The plastic work equals or exceeds the heat generated at every instant of the first quarter cycle and throughout most of the experiment as well. The stress in the first quarter cycle is considerably higher than in Figs. 1, 4, and 7 and this requires more work from the external driver. It should be emphasized that an attempt was made to prepare this specimen in the same way as the others. The hysteresis loop is more symmetrical in this specimen than in the previous ones. We have never observed significant nonhomogeneous strains in specimens having this type response during the

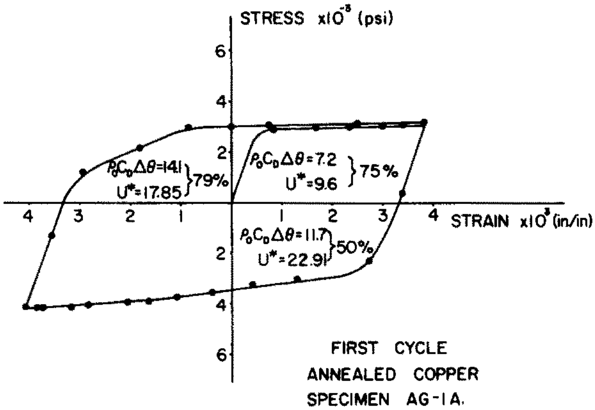

FIG. 10. Hysteresis loop for specimen No. AG-1A which is typical of specimens having the rate of plastic work exceeding the rate of temperature rise in the first quarter cycle. Note the increased symmetry of this figure with respect to Figs. 4 and 9. The amplitude of the oscillation is 0·00368. The strain is $2\varepsilon_{23}(L/2)$.

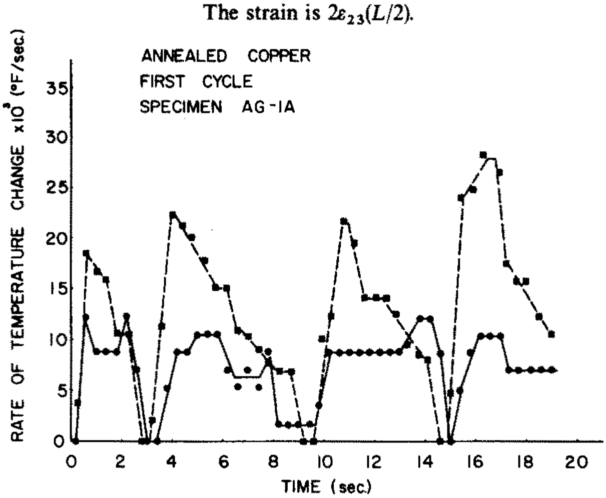

FIG. 11(a). The rate of temperature increase and the rate of doing plastic work for specimen No. AG-1A.

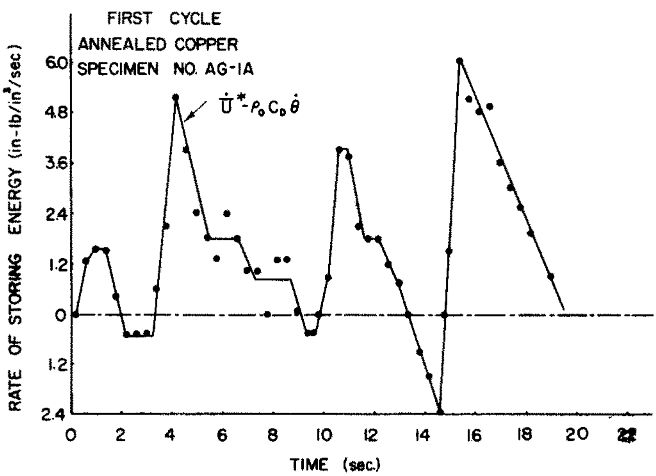

FIG. 11(b). The rate of storing energy for specimen No. AG-1A.

first quarter cycle. Subsequent hysteresis loops remain symmetrical but the center strain decreases slightly from that shown in Fig. 10. For specimen No. AG-1, *every* half cycle is such that plastic work per half cycle exceeds the heat generated. Figure 11(b) shows the rate of storing energy as a function of time for this specimen. For this particular specimen there was a general correlation between the rate of storing energy and the absolute value of the plastic strain rate, $\dot{\varepsilon}''_{23} = \dot{\varepsilon}_{23} - \dot{\sigma}_{23}/2\mu$. However, this correlation did not hold for other specimens and so the comparison is regarded as possibly accidental.

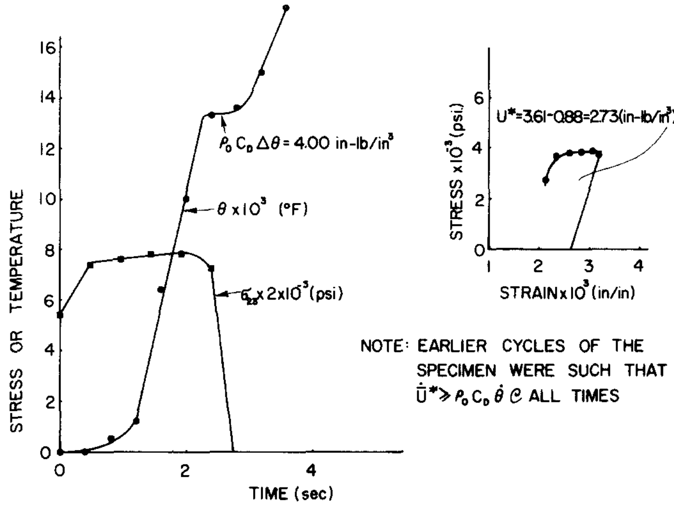

FIG. 12. The stress and temperature on restarting specimen No. AG-1 for which data are presented in Figs. 10–11(a). Note that from the time the motor started until the specimen unloads, the rate of temperature increase exceeds the work done by 4·0/2·73.

Figure 12 shows the stress and temperature for specimen No. AG-1 after twenty cycles with frequent stopping of the motor and cooling. In fact, such a cooling took place just prior to the data shown in Fig. 12. Notice that the stress in this figure is the same as developed in specimen No. AG-22-G and shown in Fig. 9. The insert in Fig. 12 contains a part of the hysteresis loop for the start up part of the cycle. As indicated in the figure, the heat generated between starting the motor and unloading exceeds the plastic work done by the ratio of 4·0/2·73. This response indicates the general difficulty of defining the reference state of the material by means of stress and strain.

A third kind of response

In the case of annealed aluminum reported in [1], data were presented for which there was a near-perfect cyclic balance of plastic work and heat generated. Some annealed copper specimens have this response as well. Typical data for this class of response are shown in Figs. 13(a) and 14. In Fig. 13(a), one sees that the rate of doing plastic work exceeds the heat generated during the time when the material is just leaving the plastic range; while later in the cycle the rate of heat generation exceeds the rate of doing plastic work. Over the first quarter cycle there is a balance of heat generated and plastic work done.

Figure 13(b) shows the rate of storing energy as a function of time. The positive rising portions of Fig. 13(b) (when $\dot{U}^* > \rho_0 C_D \dot{\theta}$) correlate reasonably well with the absolute value of the rate of plastic straining, but other parts of the curve do not. The data in Fig. 13(b) indicate that the external forces attempt to raise the material to a higher energy level during the first part of the cycle, but the material does not choose to sustain itself at this higher level.

The specimens in this class deform homogeneously and have symmetrical hysteresis loops just as in the second class of response and in contrast to the first group of specimens.

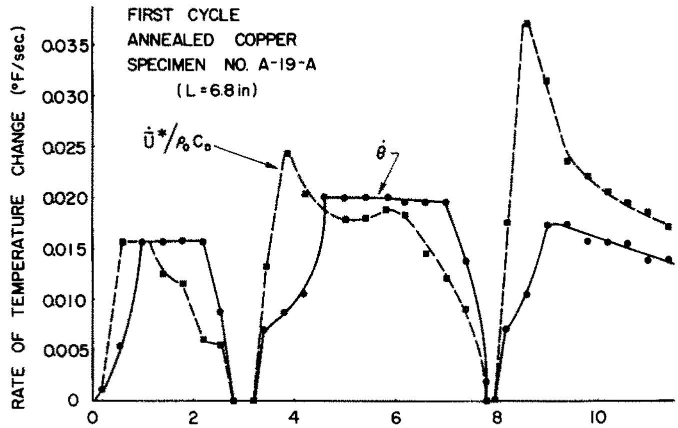

FIG. 13(a). The rate of doing plastic work and the rate of temperature rise for specimen No. A-19-A for which there is a perfect balance between motor start and initial unloading.

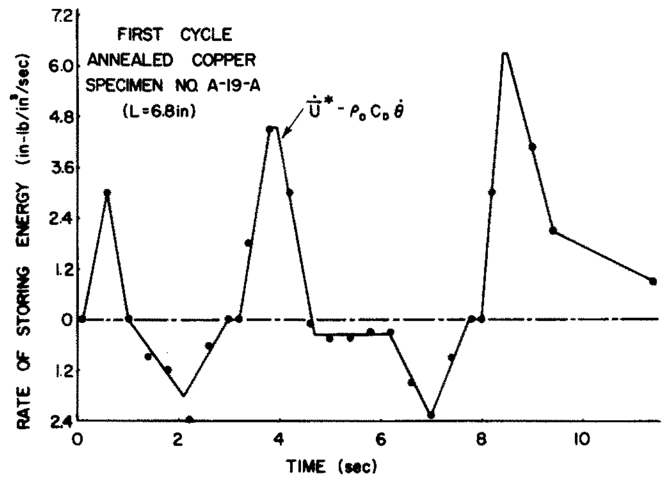

FIG. 13(b). The rate of storing energy for specimen No. A-19-A.

As received copper

Some specimens were tested in the "as received" state. These specimens are much stronger than the annealed ones and deformed homogeneously with respect to axial position. Typical data are shown in Figs. 15–17 for the first cycle and in Figs. 18 and 19 for

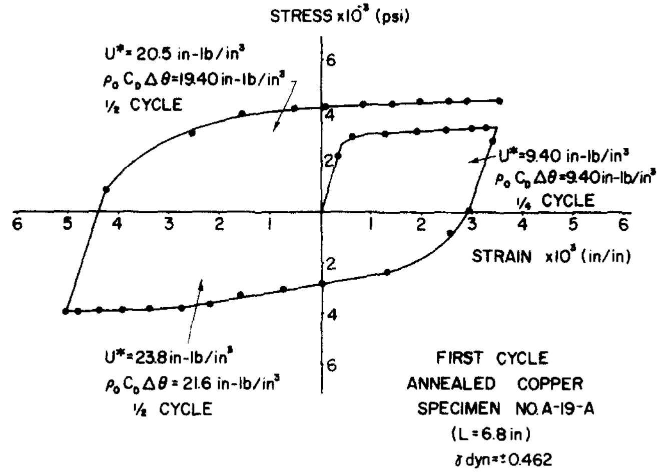

FIG. 14. The hysteresis loop for specimen No. A-19-A for which other data are presented in Fig. 13(a). The amplitude of the oscillation is 0·00462.

the fifth cycle. The motor was stopped and the specimen permitted to equilibrate with its environment prior to starting the deformation shown in Figs. 18 and 19. Notice that the hysteresis loop is wider in Fig. 19 than in Fig. 17. Furthermore, the "loading" portion of the curve through the origin does not have the original elastic modulus. Note that this specimen did not remain at constant temperature when it was "unloading". In this respect it was like Fig. 8. The response is such that small plastic strain rates exist, causing

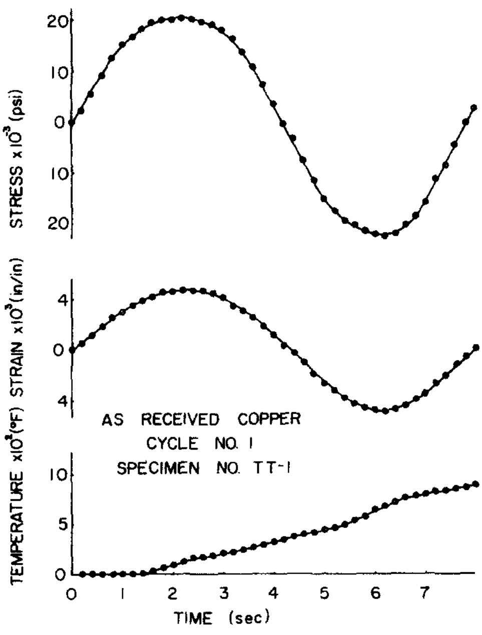

FIG. 15. Stress, strain and temperature for "as received" copper specimen No. TT-1 for an amplitude of oscillation of 0·0044.

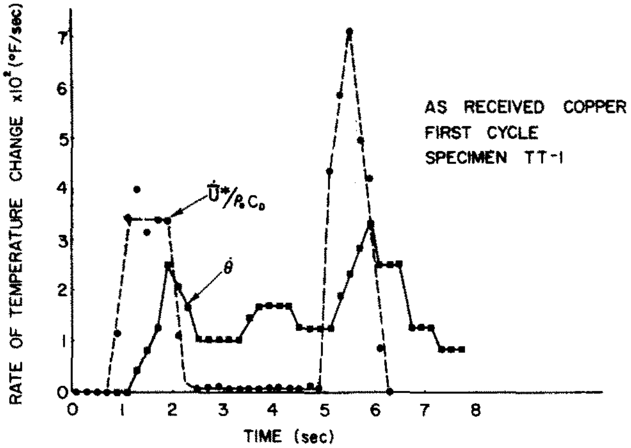

FIG. 16. The rate of temperature increase for "as received" specimen No. TT-1 (see Fig. 15).

heat to be generated during unloading portions of the cycle. There is a near perfect balance between plastic work done on the specimen and the heat that is generated during the first cycle. As shown in Fig. 18, the history of the heat generation and that of the plastic work are entirely different. Furthermore, the first half cycle shown in Fig. 18 did not raise the temperature at all, but later on in the cycle the energy reappeared as heat.

Average results

Results from twelve specimens are summarized in Table 1. The Class in this table is determined by the response in the first quarter cycle. For Class No. 1, $\rho_0 C_D \Delta\theta > U^*$; for Class No. 2, $U^* > \rho_0 C_D \Delta\theta$; for Class No. 3, $\rho_0 C_D \Delta\theta \doteq U^*$ for the first quarter cycle. It is interesting to note that the plastic work in the static test reported in Fig. 1 is $U^* = 5{\cdot}5$ in-lb/in^3 in the first quarter cycle for a strain amplitude of 0·00335, while the average of specimens with amplitude $\gamma = 0.00368$ is 6·5 in-lb/in^3.

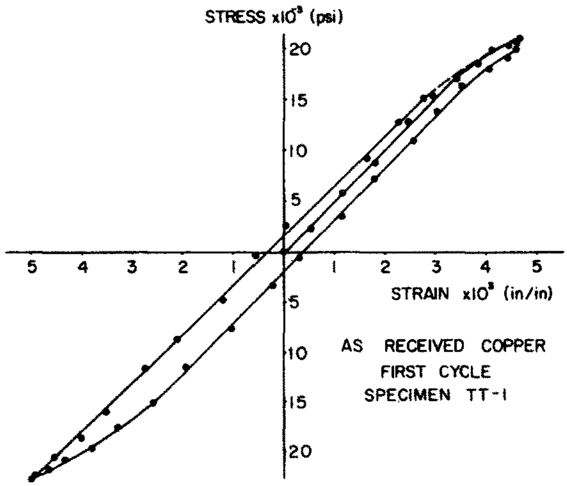

FIG. 17. The hysteresis loop for "as received" specimen No. TT-1 (see Figs. 15–16). The strain is twice ε_{23} measured at the center $x_3 = L/2$.

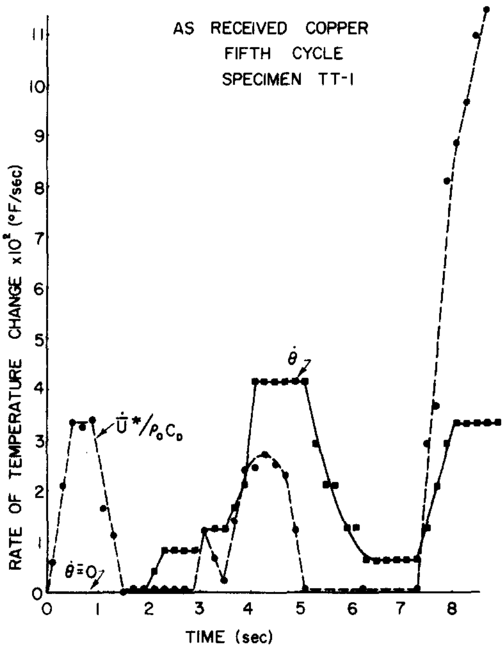

FIG. 18. The rate of temperature change and doing plastic work for the fifth cycle of specimen No. TT-1. The amplitude of the oscillation is 0·0044.

Results for specimens which have been oscillated through twenty or more cycles (except specimen No. TT-5 which is only five cycles) are shown in Table 2. The Class No. is the same one given in Table 1 for each specimen.

TABLE 1

Specimen No.	Class No.	Plastic work per ¼ cycle (in-lb/in³)	Heat generated per ¼ cycle (in-lb/in³)	% of plastic work that appears as heat
AG-1	2	9·6	7·2	75
AG-3	1	1·4	6·7	480
AG-4*	2	14·0	5·0	37
AG-6*	1	1·4	10·5	750
AG-16	2	7·0	4·5	65
AG-18†	3	9.7	9·0	92
AG-19†	3	9·4	9·4	100
AG-20†	3	9·7	9·0	92
AG-21†	3	11·6	12·3	106
AG-22	1	4·3	12·5	346
TT-5‡	1	4·7	8·2	175
Average		7·5	8·55	114
TT-1‡	A.R.	8·0	5·1	64

* Very nonhomogeneous strain.
† $\gamma = 0{\cdot}0048$.
‡ $\gamma = 0{\cdot}0045$.

TABLE 2

Specimen No.	Class No.	Plastic work per cycle	Heat generated per cycle	% of plastic work that appears as heat
AG-1	2	50·2	31·2	62
AG-3	1	56·7	34·6	61
AG-16	2	51·1	43·0	84
AG-20	3	76·6	72·0	94
AG-21	3	75·0	63·4	85
TT-3	3	69·5	62·4	90
TT-5	1	89·5	63·0	70
Average	—	66·7	52·7	79
TT-1	A.R.	36·0	36·3	101

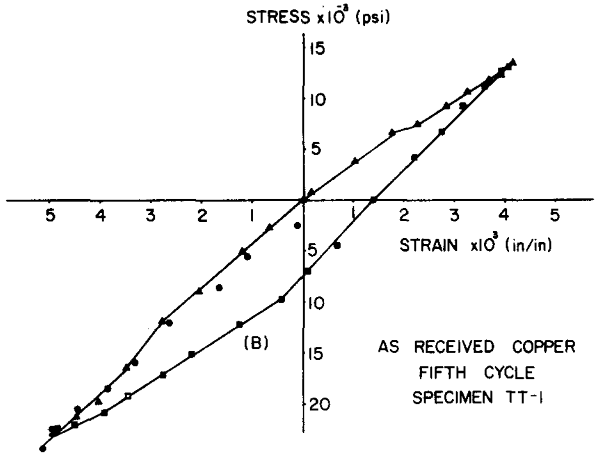

FIG. 19. The hysteresis loop for the fifth cycle of specimen No. TT-1. The amplitude of the oscillation is 0.0044. Compare with Fig. 17.

3. DISCUSSION

General

The experimental data presented above demonstrate that the small temperature changes which are produced during plastic deformation are energetically as large as the plastic work done on the specimen. It is also clear that the rate at which heat is generated can sometimes exceed the rate at which plastic work is done. Furthermore, the annealed copper requires many more oscillations to reach a "limit cycle" in stress–strain space than aluminum does. The annealed copper does not conform to the postulates used in [1] so that the coupled thermoplasticity theory developed there will not be as applicable to copper as to aluminum. The average response of several copper specimens should be approximately consistent with predictions made by the coupled heat conduction equation in [1].

It is obvious that specimens whose response is designated as Class 3 cannot be used to establish basic constitutive equations because of the nonhomogeneous deformations.

However, the very fact that such metastable configurations exist is itself significant to this author.

The experimental data presented above can be used to evaluate the rate of storing energy according to equation (8) which is also derived in the Appendix. For these calculations the elastic strain is simply taken as the linearly elastic relation, $e'_{i\alpha} = S_{i\alpha}/2\mu$. A typical history for the rate of storing work is shown in Fig. 20. Also shown in this figure is the plastic strain as a function of time. The dashed lines in the energy history are for those times when the material is unloading and the plastic strain rate is very small.

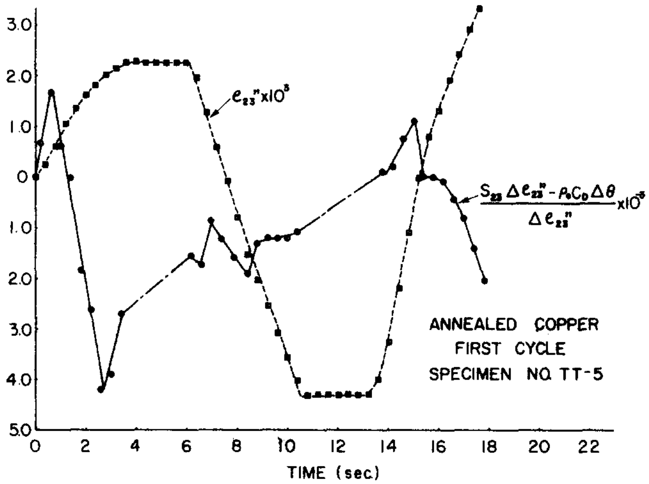

FIG. 20. The plastic strain e''_{23} and the coefficient $\rho_0 \partial U/\partial e''_{23}$ as given by equation (A15) for specimen No. TT-5.

The corresponding data for specimen AG-1A have a somewhat better correlation between the energy and the plastic strain than the data shown in Fig. 20. The point of Fig. 20 is that the energy stored is not an obviously simple relation of the plastic strain. The data shown in Figs. 21–23 are taken from figures like 20, and subsequently eliminating the

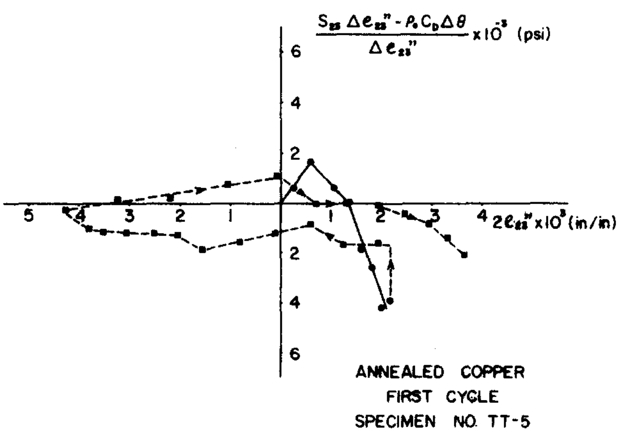

FIG. 21. The coefficient $\rho_0 \partial U/\partial''_{23}$, as given by equation (A15), expressed as function of the plastic strain for the first cycles of specimen No. TT-5.

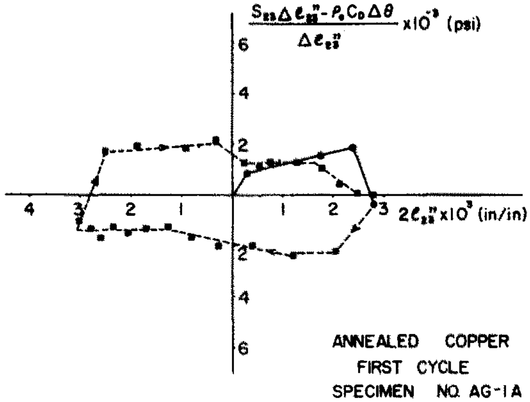

FIG. 22 The coefficient $\rho_0 \partial U/\partial e''_{23}$, as given by equation (A15), expressed as a function of the plastic strain for the first cycle of specimen No. AG-1A.

time variable. The significant features of these figures are (a) that $\partial^2 U/\partial e^{2''}_{23} > 0$ for all specimens as they leave the elastic range in the first cycle, (b) all specimens have $\partial U/\partial e''_{23} = 0$ and $\partial^2 U/\partial e^{2''}_{23} < 0$ several places in their deformation, (c) in later parts of the cycle, there is an appreciable region where $\partial U/\partial e''_{23} = \partial^2 U/\partial e^{2''}_{23} = 0$. It is possibly significant that $\partial^2 U/\partial e^{2''}_{23} > 0$ until the static "flow" stress of 2000 lb/in^2 was reached by $\rho_0 \partial U/\partial e''_{23}$.

It is entirely obvious that Figs. 21–23 do not reproduce the stress–strain relation as they would if the experiment could be considered isothermal.

It is interesting to observe that the stress in the first cycle of the oscillatory tests is not always the same as in the static test. It is also interesting to note that the values that do exist in the first cycle of the oscillatory test are approximately the same as obtained on either the first or one of the subsequent cycles of the static experiment.

Conclusion

The major conclusion of this paper is that tests being used to determine constitutive relations are not isothermal. If they are done very slowly, then heat conduction must be

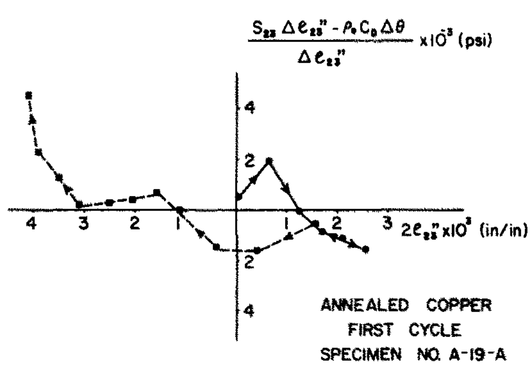

FIG. 23. The coefficient $\rho_0 \partial U/\partial e''_{23}$, as given by equation (A15), expressed as a function of the plastic strain for the first cycle of specimen No. A-19-A.

considered. It is also demonstrated that the rate of heat generation can exceed the rate of doing plastic work in an adiabatic situation. A coupled thermoplasticity theory is presented which can be made consistent with these experimental observations.

The experimental method utilized has the advantage that local aspects of storing energy can be studied. This may have some advantage in terms of interpretation of the defect structure and its effect on the response. There are certainly differences between the local response and the average over a cyclic for the storing of energy. When an annealed specimen is first deformed into the plastic range, it stores energy so that $\partial^2 U/\partial e_{i\alpha}^{2''} > 0$, for some range of plastic strains. Later in the cycle this is not the case.

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APPENDIX

THERMODYNAMICS OF COUPLED THERMOPLASTICITY

In [1], the author developed a coupled thermoplasticity theory by following the procedure given in [10] with a certain choice for the dissipation deformation variable. After seeing some recent developments [11] in continuum mechanics it occurred to him that his development in [1] did not reflect the Principle of Equipresence and that [1] could be generalized and also made more consistent. The procedure used in [11] has the additional advantage that one does not need to explicitly assign the dissipative deformation variables. The development presented here has the same basic ingredients as [12] except that it is limited, for purposes of simplicity, to cases where the small strain tensor is applicable. By incorporating some experimental data, we can derive more explicit, but less generally applicable, expressions than [12].

Equation (3) can be rewritten in terms of the free energy φ, defined by $U = \varphi + Ts$, where s is the entropy density and T is the absolute temperature. The result is

$$\rho_0 \dot{\varphi} = S_{i\alpha}\dot{e}_{i\alpha} + \sigma_{kk}\dot{\varepsilon}_{ll}/3 - \rho_0 T\dot{s} - \rho_0 \dot{T} s - \partial Q_\alpha/\partial X_\alpha \tag{A1}$$

where the work done by the stresses has been decomposed into convenient terms. The deviatoric strain tensor is further decomposed into two parts, the elastic strain, $e'_{i\alpha}$, and the plastic strain, $e''_{i\alpha}$. Hence by definition

$$e_{i\alpha} = e'_{i\alpha} + e''_{i\alpha} \tag{A2}$$

where the elastic strain is the "reversible" part of the total strain. Experimentally $\dot{e}''_{i\alpha} \geq 0$ on "unloading".

For thermodynamic purposes, dilatation, the elastic and plastic deviatoric strains are regarded as independent variables. The free energy of a plastic material is therefore a function of the form

$$\varphi = \varphi(e'_{ij}, e''_{ij}, \varepsilon_{ll}, T). \tag{A3}$$

Similarly,

$$Q_\alpha = Q_\alpha(e'_{ij}, e''_{ij}, \varepsilon_{ll}, T)$$

Equation (A1) then yields:

$$(S_{i\alpha} - \rho_0 \partial\varphi/\partial e'_{i\alpha})\dot{e}'_{i\alpha} + (S_{i\alpha} - \rho_0 \partial\varphi/\partial e''_{i\alpha})\dot{e}''_{i\alpha} + (\sigma_{kk}/3 - \rho_0 \partial\varphi/\partial\varepsilon_{kk})\dot{\varepsilon}_{ll} - \rho_0(\partial\varphi/\partial T + s)\dot{T}$$
$$-(Q_\alpha/T)(\partial T/\partial X_\alpha) = \rho_0 T\dot{s} + T\partial(Q_\alpha/T)/\partial X_\alpha \tag{A4}$$

The right-hand side of equation (A4) is the rate of entropy production and by the postulate known as the second law of thermodynamics is non-negative. By considering equation (A4) in several convenient processes,* restrictions on the functions can be deduced because of the non-negative character of the equation. These restrictions also simplify equation (13) for future calculations.

Convenient process No. 1

It is assumed hat experimental evidence exists which shows that it is possible to uniformly change he temperature of a body and that by adjusting the stresses, this produces no defoimations. Specifically, it is considered to be possible to have:

$$Q_\alpha = \dot{e}'_{i\alpha} = \dot{e}''_{i\alpha} = \dot{\varepsilon}_{ll} \equiv 0,$$

while $\dot{T}$ can have either sign. Hence the second law requires

$$s + \partial\varphi/\partial T = 0 \tag{A5}$$

Of course this is true for *all* processes since equation (A3) does not depend on the time derivatives of $e'_{i\alpha}$, $e''_{i\alpha}$, ε_{ll} and T.

* In [11] a process is considered to be a special solution of the full set of field equations. To accomplish the special solutions, a source term is included in the energy equation. The author chooses to consider a process as an idealized experiment. Obviously in an applicable theory, the two concepts of a process should lead to the same result. The advantage of regarding the convenient processes as experimental is that some ideas of the applicability of the theory can then be obtained. There are also advantages to regarding it as a solution of the field equations.

Convenient process No. 2

We will further restrict our consideration of materials for which experimental evidence shows that purely dilational deformations are elastic and for which the temperature in such a process is uniform in space. Thus, $\dot{e}'_{i\alpha} = \dot{e}''_{i\alpha} = Q_\alpha = 0$ while $\dot{\varepsilon}_{ll}$ can be changed in sign. Therefore

$$\sigma_{kk}/3 - \rho_0 \partial\varphi/\partial\varepsilon_{kk} = 0 \tag{A6}$$

for consistency with the second law.

Convenient process No. 3

It is taken to be experimentally verified that a purely deviatoric homogeneous deformation can also be uniform in temperature. At any point in the history it is assumed that the plastic strain rate can be made vanishingly small. The normal way of doing this is by "unloading"; then at any stress, the elastic strain rate can be of either sign. The second law therefore requires

$$S_{i\alpha} - \rho_0 \partial\varphi/\partial e'_{i\alpha} = 0 \tag{A7}$$

for consistency.

Convenient process No. 4

It is assumed that thermal gradients can exist in a purely elastic deformation. Hence that $\dot{e}_{i\alpha} = 0$, when Q_α does not. The second law leads to the usual condition that

$$(Q_\alpha/T)\partial T/\partial X_\alpha \leq 0. \tag{A8}$$

Convenient process No. 5

It is taken to be experimental evidence that a homogeneous plastic deformation can be such that a uniform temperature field exists. Equation (A4) and the second law then yield

$$(S_{i\alpha} - \rho_0 \partial\varphi/\partial e''_{i\alpha})\dot{e}''_{i\alpha} \geq 0 \tag{A9}$$

since $Q_\alpha = 0$.

Comment on equation (A9)

One might be tempted to approximate experimental evidence on unloading with the relation that $\dot{e}''_{i\alpha} \equiv 0$. This would establish the equality sign in equation (A9) and once this is done to conclude that

$$S_{i\alpha} - \rho_0 \partial\varphi/\partial e''_{i\alpha} = 0 \tag{A10}$$

because the relation must also be true when $\dot{e}''_{i\alpha} \neq 0$. However, this is obviously an incorrect use of this result.

General

Incorporating equations (A5) to (A7) into equation (A1) yields

$$\rho_0 T\dot{s} = -\partial Q_\alpha/\partial X_\alpha + (S_{i\alpha} - \rho_0 \partial\varphi/\partial e''_{i\alpha})\dot{e}''_{i\alpha} \tag{A11}$$

The last term in equation (A11) is more general than the development given in [1], or said differently the stress is not always the force conjugate to $e''_{i\alpha}$. The entropy rate is

$$\dot{s} = (\partial s/\partial T)\dot{T} + (\partial s/\partial \varepsilon_{kk})\dot{\varepsilon}_{kk} + (\partial s/\partial e'_{i\alpha})\dot{e}'_{i\alpha} + (\partial s/\partial e''_{i\alpha})\dot{e}''_{i\alpha} \tag{A12}$$

The last term in equation (A12) is also added to the corresponding result in [1]. Equation (A5) then yields: $\partial s/\partial T = -\partial^2\varphi/\partial T^2$ and equation (A6) $(3\rho_0)\partial s/\partial \varepsilon_{kk} = -(\partial \sigma_{kk}/\partial T)$.

Special

The considerations are now to be further limited to those materials for which

$$\sigma_{kk} = (3\lambda + 2\mu)(\varepsilon_{kk} - \alpha\theta)$$

where α is the coefficient of thermal expansion and λ and μ are Lamé constants. The elastic shear strain is now assumed to be linearly related to the stress, $S_{i\alpha} = 2\mu e'_{i\alpha}$. Experimental evidence is also quite strong in metals that this deviatoric stress–strain relation does not involve temperature. In some instances, however, the moduli are temperature dependent, ($\mu = \mu(T)$), so these materials are not covered in what follows. Equations (A5) and (A7) then imply that $\partial s/\partial e'_{i\alpha} = 0$. It is customary to define the specific heat at constant deformation C_D as

$$C_D = (1/T)\partial s/\partial T = -(1/T)\partial^2\varphi/\partial T^2 \tag{A13}$$

When these are incorporated into equation (A11), one finds that

$$(\rho_0 \partial U/\partial e''_{i\alpha})\dot{e}''_{i\alpha} = -\partial Q_\alpha/\partial X_\alpha - \alpha(3\lambda + 2\mu)T\dot{\varepsilon}_{ll}/3 + S_{i\alpha}\dot{e}''_{i\alpha} - \rho_0 C_D \dot{\theta} \tag{A14}$$

The experimental conditions used in the main text are at small times approximated by $Q_\alpha = \dot{\varepsilon}_{ll} = 0$. In this particular case, equation (A14) becomes

$$\rho_0 \partial U/\partial e''_{i\alpha}\dot{e}''_{i\alpha} = S_{i\alpha}\dot{e}''_{i\alpha} - \rho_0 C_D \dot{\theta} \tag{A15}$$

If one assumes (probably incorrectly, but frequently done) that the experiment is also isothermal, one can deduce equation (A10). For those materials for which the deviatoric constitutive equation involves no temperature, equations (A5) and (A10) combine to yield

$$\partial s/\partial e''_{i\alpha} = 0 \tag{A16}$$

which is probably nonsense. That is, the very nature of plastic deformation should alter the entropy. Thus the conclusion is that experiments in the plastic range are probably not isothermal. Said differently, if $\mu \neq \mu(T)$ and if no temperature explicitly appears in the deviatoric constitutive equation, one cannot generally neglect both Q_α and $\dot{\theta}$ in equation (A14). In fact, it is generally true [2, 5] that only a small percentage of the plastic work, $S_{i\alpha}\dot{e}''_{i\alpha}$, is stored as energy, $\rho_0 \partial U/\partial e''_{i\alpha}\dot{e}''_{i\alpha}$. Under these conditions it is probably more convenient to measure the temperature in constitutive equation experiments than it is to consider Q_α.

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Résumé—Des données expérimentales ont été présentées sur la chaleur émise durant les oscillations lentes de torsion de tubes de cuivre OFHC détrempé. Ces oscillations impliquent une amplitude d'effort d'environ $\frac{1}{2}$ pour cent pour que les déformations plastiques adviennent. La fréquence des oscillations est moins d'un quart de cycle par seconde. L'on a démontré qu'il existe un moment durant lequel des conditions presque adiabatiques existent et toutes les observations se font durant cette période. L'on a également démontré que le taux du travail plastique excède parfois le taux de la chaleur émise et vice-versa. Sur un cycle complet le travail excède toujours la chaleur émise, mais localement ceci n'est pas vérifié. Des spécimens qui émittent plus de chaleur que le travail plastique effectué sur eux ont des ventres d'hystéresis non symétriques et se déforment parfois d'une façon non homogène durant le premier cycle. Tous les spécimens ont besoin de plus de travail plastique qu'ils n'émettent en tant que chaleur quand ils quittent la région élastique durant le premier cycle.

Une théorie thermoplastique accouplée, conforme aux données expérimentales est également présentée en appendice.

Zusammenfassung—Versuchsangaben der Wärme, erzeugt während langsamer Drehschwingungen von ausgeglühten OFHC Kupferröhren sind gegeben. Diese Schwingungen bedingen eine Verterrungsamplitude von ungefähr $\frac{1}{2}$ prozent, so das plastische Verformungen vorkommen. Die Häufigkeit von Schwingungen ist weniger als ein Viertel Periode pro Sekunde. Eine Periode wird nachgewiesen, wenn beinahe adiabatische Bedingungen vorhanden sind und alle Beobachtungen werden zu dieser Zeit gemacht. Es wird nachgewiesen, das die Schnelligkeit der plastischen Arbeitsverrichtung manchesmal die Schnelligkeit der Wärmeerzeugung überschreitet, und umgekehrt. Über eine vollständige Periode überschreitet die Arbeit immer die erzeugte Wärme, aber örtlich ist das nicht der Fall. Proben, welche mehr Wärme auslösen als die plastische Arbeit verrichtet an ihnen, haben unsymmetrische Hysteresisschleifen und manchesmal deformieren ungleichartig in der ersten Periode. Alle Proben benötigen mehr plastische Arbeit als sie Wärme auslösen wenn sie den elastischen Bereich in der ersten Periode verlassen.

Eine gekoppelte thermoplastische Theorie welche mit den Versuchsangaben übereinstimmt ist ebenfalls in einem Anhang präsentiert.

Абстракт—Даются экспериментальные данные тепла, получающегося во время медленных крутильных колебаний отожжённых OFHC медных трубок. Эти колебания вовлекают деформацию амплитуды приблизительно на пол процента, так что происходят пластические деформации. Частота колебаний меньше четверти круга в секунду. Показано, что есть период, когда существуют почти адиабатические условия и все наблюдения производились в это время. Показано, что быстрота выделки пластической работы иногда превосходит скорость образования тепла и наоборот. В течение полного цикла работа всегда превосходит образующееся тепло, но местно—это не так. Образцы, которые освобождают больше тепла, чем пластическая работа, которая производится на них, имеют несимметрические петли запаздывания и иногда негомогенно деформируются в первом цикле. Все образы требуют больше пластической работы, чем то, что они освобождают в виде тепла, когда оставляют эластический район в первом цикле.

Теория соединённой термопластики, которая согласуется с опытными данными также представлена в дополнении.